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REMARK ON THE APPROXIMATE SOLUTION OF THE PROBLEMS OF LINEAR VISCOELASTICITY

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It is proved that for a broad class of quasistatic problems the correspondence principle is a corollary of the Volterra principle, and a foundation is given for the Il'iushin method of approximation [15, 16].

1. It is known that the solution, based on the Volterra principle [1, 2] for the quasistatic problem of linear viscoelasticity for a homogeneous anisotropic material, whose elastic properties are invariant relative to the time reference point, reduces to the construction of some analytic function of several Volterra operators with kernels dependent on the difference in the arguments. It is shown below that under quite general assumptions the exact solution of this problem can be obtained in closed form.

Let us assume that the elastic solution is

$$q(x) = (A(\omega_1, \ldots, \omega_n)\, r)(x), \quad r = r(x), \quad q = q(x), \quad x \in \Phi \tag{1.1}$$

Here Φ is some domain of n-dimensional space ($n = 1, 2, 3$); r and q are elements of the Banach space $L^{(p)}(\Phi)$ $(1 \leqslant p \leqslant \infty)$; $A(\omega_1, \ldots, \omega_n)$ is a linear operator acting in this space and analytically dependent on the elastic constants $\omega_1, \ldots, \omega_n$.

Let us set

$$\omega_i = \omega_i^{(0)}(1 - z_i) \qquad (i = 1, 2, \ldots, n)$$

$$f(z) = f(z_1, \ldots, z_n) = A(\omega_1^{(0)}(1 - z_1), \ldots, \omega_n^{(0)}(1 - z_n)) \tag{1.2}$$

According to the Volterra principle, the viscoelastic solution is obtained from the elastic solution by replacing z_i by the appropriate viscoelastic operators H_i with continuous or slightly singular kernels

$$(H_i u)(t) = \int_0^t H_i(t - \tau)\, u(\tau)\, d\tau \tag{1.3}$$

In connection with the fact that the Volterra operators H_i (1.3) should be substituted in an operator function rather than in a scalar function, it is here impossible to use the apparatus of multidimensional operational calculus [3] developed for scalar functions. It is known that for the boundedness of the operator H_i in (1.3) in the space $L_\beta^\infty(0, \infty)$ of functions $u(t)$ measurable in $(0, \infty)$ with the norm

$$\| u \|_\beta = \operatorname*{ess\,sup}_{0 \leqslant t < \infty} | u(t) |\, e^{-\beta t} \tag{1.4}$$

the condition

$$\| H_i \|_\beta = \int_0^\infty | H_i(t) |\, e^{-\beta t}\, dt < \infty \tag{1.5}$$

is necessary and sufficient. Let us assume that the condition (1.5) is satisfied for $i = 1, 2, \ldots, n$ and that the operator function $f(z)$ in (1.2) is analytic in the polycircle D

$$D\{z_i;\ |z_i| \leqslant \|H_i\|_\beta + \Delta_i \quad (i = 1, 2, \ldots, n)\} \tag{1.6}$$

where Δ_i are some positive numbers. It hence follows that the expansion [4]

$$f(z) = \sum_\alpha \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha \tag{1.7}$$

$$z = (z_1, \ldots, z_n), \quad \alpha = (\alpha_1, \ldots, \alpha_n), \quad z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}, \quad 0 = (0, \ldots, 0)$$

$$\alpha! = \alpha_1! \ldots \alpha_n!, \qquad \partial^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots\cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}$$

which converges in the operator norm generated by the topology of the space $L^{(p)}(\Phi)$, holds in D.

Let us introduce the Banach space $\Lambda_\beta^{(p,\infty)}$ of vector functions $u(t)$ with values in $L^{(p)}(\Phi)$, measurable and bounded almost everywhere on $0 \leqslant t < \infty$ [5] with the norm (1.4). Here and everywhere henceforth, $|a|$ ($\|A\|$) denotes the norm of the element (the operator A) in $L^{(p)}(\Phi)$, $\|a\|_\beta$ ($\|A\|_\beta$) is the norm in $\Lambda_\beta^{(p,\infty)}$.

By definition, let us set

$$f(H) = f(H_1, \ldots, H_n) = \sum_\alpha \frac{\partial^\alpha f(0)}{\alpha!} H^\alpha \tag{1.8}$$

$$H^\alpha = H_1^{\alpha_1} \ldots H_n^{\alpha_n}$$

Volterra [6] used precisely such a construction substantially in the problem of the equilibrium of viscoelastic sphere on whose surface the displacements are given (see also [2]). The series (1.8) converges in the operator norm generated by the topology of the space $\Lambda_\beta^{(p,\infty)}$. It follows from (1.8) that

$$f(H) = f(0) + G \tag{1.9}$$

Here G is a Volterra operator acting in $\Lambda_\beta^{(p,\infty)}$

$$(Gu)(t) = \int_0^t G(t-\tau)\, u(\tau)\, d\tau \tag{1.10}$$

with the kernel determined by the expansion

$$G(t) = \sum_{\alpha \neq 0} \frac{\partial^\alpha f(0)}{\alpha!} H^{(\alpha)}(t), \quad H^{(\alpha)}(t) = H_1^{(\alpha_1)}(t) * \ldots * H_n^{(\alpha_n)}(t) \tag{1.11}$$

$$H^{(\alpha_i)}(t) = H_i(t) * \cdots * H_i(t)$$

where the asterisk denotes the convolution, and the number of factors in the last equation agrees with the superscript. Since (1.11) converges in the mean with weight e^{-3t} on a half-axis, then

$$\int_0^\infty |G(t)|\, e^{-3t}\, dt < \infty \tag{1.12}$$

In order to obtain a rule for evaluating $G(t)$, let us apply a Laplace time transformation to (1.11) term by term. We obtain

$$g(w) = \sum_{\alpha \neq 0} \frac{\partial^\alpha f(0)}{\alpha!} h^\alpha(w), \quad \operatorname{Re} w \geqslant 3, \quad h^\alpha(w) = h_1^{\alpha_1}(w)\, h_2^{\alpha_2}(w) \cdots h_n^{\alpha_n}(w) \tag{1.13}$$

$$h_i(w) = \int_0^\infty H_i(t)\, e^{-wt}\, dt, \quad g(w) = \int_0^\infty G(t)\, e^{-wt}\, dt$$

The rule obtained is the known correspondence principle [7]. Therefore, under the assumptions introduced relative to the elastic solution (1.2), the correspondence principle is a corollary of the Volterra principle. The reverse is also true.

The application of the Volterra principle sometimes results in the calculation of a scalar function $f(z)$ of the operators H_i in (1.3) [8 – 11]. In this case the apparatus of the multidimensional operational calculus can be used in normed rings [3]. This permits weakening of the demand imposed above on the function $f(z)$, namely, it is sufficient to assume that the function $f(z)$ is analytic in the set S

$$S\{z_i,\ z_i = h_i(w),\quad \mathrm{Re}\, w \geqslant \beta \quad (i = 1, 2, \ldots, n)\}$$

which is narrower than the polycircle D in (1.6).

The operators H_i in (1.3) can be identified with elements of the maximal ideal M_∞ of a commutative normed ring $V_+^{\langle e^{-\beta t}\rangle}$ [3]. The set S is the combined spectrum of the elements H_i in this ring.

According to a theorem of Shilov-Arens-Calderon [3], there exists an element $G(t)$ of the ring $V_+^{\langle e^{-\beta t}\rangle}$ such that equality

$$g(w) = f(h(w)) - f(0),\ f(h(w)) = f(h_1(w))_1, \ldots, h_n(w)) \tag{1.14}$$

holds for its Laplace transform $g(w)$. This element belongs to the maximum ideal M_∞ and therefore has the form (1.10) and satisfies the condition (1.12). According to the general aspects of multidimensional operational calculus in normed rings, (1.9) follows from (1.14) (with $f(0)$ replaced by $f(0)\, I$). Therefore, we have again arrived at the correspondence principle [7]. Let us note that the above remains valid for locally analytic normal multivalued functions [12], particularly for roots and logarithms.

The papers [13, 14] are devoted to scalar analytic functions of a Volterra operator of the form (1.3). However, the relationship between the correspondence and Volterra principles has not been established therein.

2. In applications, an approximate solution which is obtained as a result of replacing the elastic solution (1.2) by its approximating polynomial or some other simple function $P(z)$ of the elastic constants, is often sought instead of the exact solution of the viscoelastic problem. Such is the procedure when the elastic solution is awkward or its analytical form is unknown [15, 16].

Let us show that if the operator-function $f(z)$ in (1.2) is analytic in the domain D in (1.6), then the functional analytic approach affords the possibility of giving this approximation method a foundation. A foundation for the approximation method of [15, 16] is simultaneously obtained.

Let D' be the polycircle $\{z_i;\ |z_i| \leqslant \|H_i\|_0 + \Delta_i'\ \ (i = 1, 2, \ldots, n)\}$, where $0 < \Delta_i' < \Delta_i$, and $\partial_0 D'$ is its skeleton [4]

$$d = \max_{\partial_0 D'} |f(z) - P(z)| \tag{2.1}$$

Let us use the following integral representations from (1.6) and (1.7), which are well known for the case of the scalar function $f(z)$ [3, 17]:

$$f(H) = \left(\frac{i}{2\pi}\right)^n \int\limits_{\partial_0 D'} f(z_1, \ldots, z_n) \prod_{i=1}^{n} (H_i - z_i I)^{-1}\, dz_1 \cdots dz_n \tag{2.2}$$

From (2.1), (2.2) and from the estimate [18]

$$\|(H_i - z_i I)^{-1}\|_0 \leqslant (|z_i| - \|H_i\|_0)^{-1} = (\Delta_i')^{-1} \quad \text{for} \quad |z_i| \leqslant \|H_i\|_0 + \Delta_i'$$

we obtain

$$\|f(H) - P(H)\|_0 \leqslant d \prod_{i=1}^{n} (\|H_i\|_0 + \Delta_i')(\Delta_i')^{-1} \tag{2.3}$$

It follows from (2.3) that the viscoelastic solution $f(H)$ can be approximated, with any a p r i o r y assigned accuracy, by a polynomial of the operators $H_1, \ldots, H_n$, since a partial sum of the series (1.7) can be taken as $P(z)$, say. Such a polynomial approximation of the elastic solution (for $n = 1$) was used (without foundation) in [10] in examining a particular problem,

3. The need to construct n-operator functions of the form

$$(W_i u)(t) = \int_{-\infty}^{t} H_i(t - \tau) u(\tau) d\tau \tag{3.1}$$

arises in the study of damping properties of elastically hereditary systems [19, 20].

Let us consider these operators in the space $L_\beta^\infty(-\infty, \infty)$ of the functions $u(t)$, which are measurable on the whole number axis with the norm

$$\|u\|_\beta = \underset{-\infty<t<\infty}{\text{ess} \quad \sup} |u(t)| e^{-\beta t}$$

The condition (1.5) is necessary and sufficient for boundedness of the operator W_i (3.1) in $L_\beta^{(\infty)}(-\infty, \infty)$. The product of two operators of the kind W_i is an operator of the same kind, whose kernel is the convolution of kernels of the cofactors. It hence follows that all that has been mentioned above in Sects. 1 and 2 remains valid upon replacement of H_i from (1.3) by W_i from (3.1).

4. The approximation method of [15, 16] is that the elastic solution, considered as a function of the elastic constants, is approximated by some simple function of these constants, and the correspondence principle is applied in the sense of [7] to the approximating function. Since the correspondence principle is a corollary of the Volterra principle, the operator viewpoint can be used in examining the approximation method of [15, 16]. Hence, the estimate (2.3) (under the assumption that the conditions under which it can be obtained are satisfied) yields a foundation for this approximation method.

Let us note that for $n = 1$ the assumption that the kernel of the Volterra operator depends on the difference between the arguments can be discarded, and an operator H of the form

$$(Hu)(t) = \int_0^t H(t, \tau) u(\tau) d\tau, \quad \|H\|_0 = \sup_{0 \leqslant t < \infty} \int_0^t |H(t, \tau)| d\tau < \infty$$

can be examined. The estimate (2.3) remains valid even in this case.

In connection with the above, the transfer of the results obtained in [21 – 23] on the Cosserat spectrum to the case of an anisotropic medium is of interest.

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